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2-Cocycles on the Algebra of Differential Operators*

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1. INTRODUCTION

The central extension method is often used in the theory of structures and representations of Kac–Moody algebras. By central extension, we can construct many infinite-dimensional Lie algebras, such as affine Lie algebras, infinite-dimensional Heisenberg algebras, and Virasoro algebras, which have a profound mathematical and physical background [1, 5, 7,8]. We can describe the structures and some of the representations of these Lie algebras by using this method [4, 9].

If $\mathfrak{G}$ is a Lie algebra, it is well known that all 1-dimensional central extensions of $\mathfrak{G}$ are determined by the 2-cocycle group of $\mathfrak{G}$. If a 2-cocycle is induced by a linear function on $\mathfrak{G}$, this 2-cocycle is called trivial; in this case, the 1-dimensional central extension is also trivial, i.e., it is isomorphic to $\mathfrak{G} \oplus \mathbb{C}c$ as a direct sum of ideals. If we want to describe all 1-dimensional central extensions of $\mathfrak{G}$, it is necessary to determine all non-trivial 2-cocycles or the second cohomology group of $\mathfrak{G}$ [3].

Let $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomial ring over a complex field $\mathbb{C}$, $\alpha_1(t^m, t^n) = m\delta_{m,-n}$, $m, n \in \mathbb{Z}$. We can construct a 1-dimensional central extension of $\mathbb{C}[t, t^{-1}]$ by using α_1, and the Lie bracket on $\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ is defined by

$$[t^m, t^n] = \alpha_1(t^m, t^n)\, c,$$

$$[c, t^n] = 0, \qquad [c, c] = 0, \qquad m, n \in \mathbb{Z}.$$

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This is an infinite-dimensional Lie algebra. The subalgebra

$$\bigoplus_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \mathbb{C}t^n \oplus \mathbb{C}c$$

is an infinite-dimensional Heisenberg algebra.

Denote the differential operator d/dt on $\mathbb{C}[t, t^{-1}]$ by D. Let $\mathfrak{L} = \{P(t)\, D \mid P(t) \in \mathbb{C}[t, t^{-1}]\}$ be the algebra of all derivations on $\mathbb{C}[t, t^{-1}]$,

$$\alpha_2(t^{i+1}D, t^{j+1}D) = \tfrac{1}{12}(i^3 - i)\, \delta_{i,-j}, \qquad i, j \in \mathbb{Z}.$$

We can construct a 1-dimensional central extension of $\mathfrak{L}$ by using α_2, and the Lie bracket on $\mathfrak{L} \oplus \mathbb{C}c$ is defined by

$$[t^{i+1}D, t^{j+1}D] = (j-i)\, t^{i+j+1}D + \alpha_2(t^{i+1}D, t^{j+1}D)\, c,$$
$$[c, t^{i+1}D] = 0, \qquad [c, c] = 0, \qquad i, j \in \mathbb{Z}.$$

This is an infinite-dimensional Lie algebra and is called a Virasoro algebra.

Let $\mathfrak{a} = \mathbb{C}[t, t^{-1}, D]$ be the algebra of differential operators on $\mathbb{C}[t, t^{-1}]$,

$$\alpha_3(t^{l+m}D^l, t^{l'+m'}D^{l'}) = \delta_{m,-m'}(-1)^l\, l!\, l'! \begin{pmatrix} m+l \\ l+l'+1 \end{pmatrix},$$

$l \geqslant 0$, $l' \geqslant 0$, $m, m' \in \mathbb{Z}$. We can construct a 1-dimensional central extension of $\mathfrak{a}$ by using α_3, and the Lie bracket on $\mathfrak{a} \oplus \mathbb{C}c$ is defined by

$$\begin{aligned}[t^{l+m}D^l, t^{l'+m'}D^{l'}] = &\sum_{i=0}^{l} \binom{l}{i} t^{l+m}(D^{l-i}(t^{l'+m'}))\, D^{l'+i} \\ &- \sum_{j=0}^{l'} \binom{l'}{j} t^{l'+m'}(D^{l'-j}(t^{l+m}))\, D^{l+j} \\ &+ \alpha_3(t^{l+m}D^l, t^{l'+m'}D^{l'})\, c,\end{aligned}$$
$$[c, t^{l+m}D^l] = 0, \qquad [c, c] = 0, \qquad l \geqslant 0, l' \geqslant 0, m, m' \in \mathbb{Z}.$$

Generally, let us consider the algebra of differential operators $\mathfrak{A} = \mathfrak{a} \otimes gl_n(\mathbb{C})$ on $\mathbb{C}[t, t^{-1}] \otimes gl_n(\mathbb{C})$, and set

$$\alpha_4(t^{l+m}D^l \otimes A, t^{l'+m'}D^{l'} \otimes A') = \delta_{m,-m'}(-1)^l\, l!\, l'! \begin{pmatrix} m+l \\ l+l'+1 \end{pmatrix} \operatorname{tr} AA',$$

$l \geqslant 0$, $l' \geqslant 0$, $m, m' \in \mathbb{Z}$, $A, A' \in gl_n(\mathbb{C})$. We can construct a 1-dimensional central extension of $\mathfrak{A}$ by using α_4, and the Lie bracket is defined by

$$\begin{aligned}
&[t^{l+m}D^l \otimes A, t^{l'+m'}D^{l'} \otimes A'] \\
&= \sum_{i=0}^{l} \binom{l}{i} t^{l+m}(D^{l-i}(t^{l'+m'}))\, D^{l'+i} \otimes AA' \\
&\quad - \sum_{j=0}^{l'} \binom{l'}{j} t^{l'+m'}(D^{l'-j}(t^{l+m}))\, D^{l+j} \otimes A'A \\
&\quad + \alpha_4(t^{l+m}D^l \otimes A, t^{l'+m'}D^{l'} \otimes A')\, c,
\end{aligned}$$

$l \geqslant 0$, $l' \geqslant 0$, $m, m' \in \mathbb{Z}$, $A, A' \in gl_n(\mathbb{C})$.

It is easy to see that a non-trivial 1-dimensional central extension of $\mathbb{C}[t, t^{-1}]$ is not unique. All non-zero 2-cocycles on $\mathbb{C}[t, t^{-1}]$ are non-trivial and the second cohomology group is an infinite-dimensional vector space. It is known [2] that every 2-cocycle on $\mathfrak{L}$ is equivalent to a multiple of α_2. In the lectures at the Graduate School, Academia Sinica, in 1985, V. G. Kac proposed the following open problem: "Is every 2-cocycle on $\mathfrak{a}$ equivalent to a multiple of α_3?" Later, Kac proposed a more general problem [6]: "Is every 2-cocycle on $\mathfrak{A}$ equivalent to a multiple of α_4?"

In this paper, we give these problems a positive answer.

2. Basic Concepts and Main Results

We assume that the underlying field is complex field $\mathbb{C}$, though all results hold for all fields of characteristic 0.

Let $\mathfrak{G}$ be a Lie algebra. A 2-cocycle on $\mathfrak{G}$ is a bilinear $\mathbb{C}$-valued form ψ satisfying the following conditions:

(i) $\psi(a, b) = -\psi(b, a)$,

(ii) $\psi([a, b], c) + \psi([b, c], a) + \psi([c, a], b) = 0$,

for $a, b, c \in \mathfrak{G}$.

If f is a linear function on $\mathfrak{G}$, we define

$$\alpha_f(x, y) = f([x, y])$$

for $x, y \in \mathfrak{G}$. One easily checks that this is a 2-cocycle. This 2-cocycle is called a trivial 2-cocycle. A 2-cocycle φ is equivalent to a 2-cocycle ψ, if $\varphi - \psi$ is trivial.

Given a 2-cocycle α on $\mathfrak{G}$, we can construct a central extension of $\mathfrak{G}$. If the Lie bracket on $\mathfrak{G}$ is $[\,,\,]$, we define a Lie bracket $[\,,\,]_0$ on $\mathfrak{G} \oplus \mathbb{C}c$ as

$$[x + \lambda c, y + \mu c]_0 = [x, y] + \alpha(x, y)\, c, \qquad x, y \in \mathfrak{G}, \lambda, \mu \in \mathbb{C}.$$

It is easy to check that $\mathfrak{G}\oplus\mathbb{C}c$ is a Lie algebra with this bracket and every 1-dimensional central extension of $\mathfrak{G}$ can be obtained in this way. Denote this central extension of $\mathfrak{G}$ by $\mathfrak{G}(\alpha)$.

Let γ be a trivial 2-cocycle induced by f and let α be a 2-cocycle; then

$$x \mapsto x+f(x)\,c,$$
$$c \mapsto c,$$

$x\in\mathfrak{G}$, gives an isomorphism from $\mathfrak{G}(\alpha)$ to $\mathfrak{G}(\alpha+\gamma)$.

Let $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomial ring over $\mathbb{C}$, $D=d/dt$ the diffeential operator on $\mathbb{C}[t, t^{-1}]$, and $\mathfrak{a}=\mathbb{C}[t, t^{-1}, D]$ the algebra of differential operators on $\mathbb{C}[t, t^{-1}]$. $\mathfrak{a}$ has a basis $\{t^{l+m}D^l \mid l\geqslant 0,\ m\in\mathbb{Z}\}$, and

$$\begin{aligned}[][t^{l+m}D^l, t^{l'+m'}D^{l'}] &= (t^{l+m}D^l)\circ(t^{l'+m'}D^{l'})-(t^{l'+m'}D^{l'})\circ(t^{l+m}D^l)\\ &= \sum_{i=0}^{l}\binom{l}{i}t^{l+m}(D^{l-i}(t^{l'+m'}))\,D^{l'+i}\\ &\quad -\sum_{j=0}^{l'}\binom{l'}{j}t^{l'+m'}(D^{l'-j}(t^{l+m}))\,D^{l+j}.\end{aligned}$$

In particular,

$$[t^{l+m}D^l, D] = -(l+m)\,t^{l+m-1}D^l, \tag{2.1}$$

$$[t^{-1}D^l, tD] = (l+1)\,t^{-1}D^l. \tag{2.2}$$

From (2.1), (2.2) we have

$$[\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}. \tag{2.3}$$

Let $gl_n(\mathbb{C})$ be an $n\times n$ matrix algebra over $\mathbb{C}$, and $\mathfrak{A}=\mathfrak{a}\otimes gl_n(\mathbb{C})$ the algebra of differential operators on $\mathbb{C}[t, t^{-1}]\otimes gl_n(\mathbb{C})$.

Define a $\mathbb{C}$-valued bilinear form α on $\mathfrak{a}$:

$$\alpha(t^{l+m}D^l, t^{l'+m'}D^{l'}) = \delta_{m,-m'}(-1)^l\,l!\,l'!\binom{m+l}{l+l'+1}, \tag{2.4}$$

$l\geqslant 0$, $l'\geqslant 0$, $m, m'\in\mathbb{Z}$. From [5], α is a 2-cocycle on $\mathfrak{a}$.

Define a $\mathbb{C}$-valued bilinear form ψ on $\mathfrak{A}$:

$$\begin{aligned}&\psi(t^{l+m}D^l\otimes A, t^{l'+m'}D^{l'}\otimes A')\\ &\quad = \delta_{m,-m'}(-1)^l\,l!\,l'!\binom{m+l}{l+l'+1}\operatorname{tr} AA',\end{aligned} \tag{2.5}$$

$l \geqslant 0, l' \geqslant 0$, m, $m' \in \mathbb{Z}$, $A, A' \in gl_n(\mathbb{C})$. From [8], ψ is a 2-cocycle on $\mathfrak{A}$. V. G. Kac proposed the following problems:

PROBLEM 1. Is every 2-cocycle on $\mathfrak{a}$ equivalent to a multiple of α?

PROBLEM 2. Is every 2-cocycle on $\mathfrak{A}$ equivalent to a multiple of ψ?

We give these problems a positive answer.

THEOREM 2.1. *Every 2-cocycle on $\mathfrak{a}$ is equivalent to a multiple of α.*

THEOREM 2.2. *Every 2-cocycle on $\mathfrak{A}$ is equivalent to a multiple of ψ.*

3. PROOF OF THEOREM 2.1

Now we can solve Kac's first problem.
From (2.4) we have

$$\begin{aligned}&\alpha(t^{l+m}D^l, t^{l'+m'}D^{l'})=0\\ &\Leftrightarrow m\neq -m' \text{ or } m=-m', \qquad 0\leqslant m+l\leqslant l+l'.\end{aligned} \tag{3.1}$$

Let β be a 2-cocycle on $\mathfrak{a}$. We define a linear function f_β on $\mathfrak{a}$:

$$f_\beta(t^{l+m-1}D^l)=-\frac{1}{l+m}\beta(t^{l+m}D^l, D), \qquad l+m\neq 0,$$

$$f_\beta(t^{-1}D^l)=\frac{1}{l+1}\beta(t^{-1}D^l, tD).$$

Then $\beta_1=\beta-\alpha_{f_\beta}$ is a 2-cocycle on $\mathfrak{a}$ and equivalent to β. From (2.1), (2.2) we have

$$\beta_1(t^{l+m}D^l, D)=0, \qquad l+m\neq 0, \tag{3.2}$$

$$\beta_1(t^{-1}D^l, tD)=0. \tag{3.3}$$

LEMMA 3.1. $\beta_1(t^{l+m}D^l, D)=0, l\geqslant 0, m\in\mathbb{Z}$.

Proof. From (3.2) it is sufficient to consider the case $l+m=0$. Since

$$[D^l, tD]=D^l\circ(tD)-tD^{l+1}=lD^l,$$

$$\begin{aligned}\beta_1(D^l, D)&=\beta_1(D^l, [D, tD])\\ &=\beta_1([D^l, D], tD)+\beta_1(D, [D^l, tD])\\ &=-l\beta_1(D^l, D),\end{aligned}$$

this implies

$$\beta_1(D^l, D) = 0.$$

LEMMA 3.2. $\beta_1(1, \mathfrak{a}) = 0.$

Proof.

$$\begin{aligned}\beta_1(1, \mathfrak{a}) &= \beta_1(1, [\mathfrak{a}, \mathfrak{a}]) \\ &= \beta_1([1, \mathfrak{a}], \mathfrak{a}) + \beta_1(\mathfrak{a}, [1, \mathfrak{a}]) \\ &= 0.\end{aligned}$$

PROPOSITION 3.3. *If $l + m \neq -1$, then*

$$\beta_1(t^{l+m}D^l, t^{l'+m'}D^{l'}) = -\frac{l'+m'}{l+m+1}\beta_1(t^{l+m+1}D^l, t^{l'+m'-1}D^{l'}).$$

Proof. $[D, t^sD^u] = D \circ (t^sD^u) - t^sD^{u+1} = st^{s-1}D^u$. If $l + m \neq -1$, by Lemma 3.1 we have

$$\begin{aligned}\beta_1(t^{l+m}D^l, t^{l'+m'}D^{l'}) &= \beta_1\left(\frac{1}{l+m+1}[D, t^{l+m+1}D^l], t^{l'+m'}D^{l'}\right) \\ &= \frac{1}{l+m+1}\beta_1([D, t^{l'+m'}D^{l'}], t^{l+m+1}D^l) \\ &\quad + \frac{1}{l+m+1}\beta_1(D, [t^{l+m+1}D^l, t^{l'+m'}D^{l'}]) \\ &= -\frac{l'+m'}{l+m+1}\beta_1(t^{l+m+1}D^l, t^{l'+m'-1}D^{l'}).\end{aligned}$$

LEMMA 3.4. $\beta_1(t^{l+m}D^l, tD) = 0$, $l \geqslant 0$, $m \in \mathbb{Z}$.

Proof. From (3.3) it is sufficient to prove the lemma for the case $l + m \neq -1$. In this case, by Proposition 3.3 we have

$$\beta_1(t^{l+m}D^l, tD) = -\frac{1}{l+m+1}\beta_1(t^{l+m+1}D^l, D) = 0.$$

PROPOSITION 3.5. *If $m \neq -m'$* or *$m = -m'$, $0 \leqslant m + l \leqslant l + l'$, then*

$$\beta_1(t^{l+m}D^l, t^{l'+m'}D^{l'}) = 0$$

(cf. (3.1)).

Proof. $[tD, t^{l+m}D^l] = mt^{l+m}D^l$.
If $m \neq 0$, we have

$$\begin{aligned}\beta_1(t^{l+m}D^l, t^{l'+m'}D^{l'}) &= \beta_1\left(\frac{1}{m}[tD, t^{l+m}D^l], t^{l'+m'}D^{l'}\right)\\ &= \frac{1}{m}\beta_1([tD, t^{l'+m'}D^{l'}], t^{l+m}D^l) \qquad \text{(Lemma 3.4)}\\ &= -\frac{m'}{m}\beta_1(t^{l+m}D^l, t^{l'+m'}D^{l'}).\end{aligned}$$

Thus, if $m \neq 0$, $m \neq -m'$, we have

$$\beta_1(t^{l+m}D^l, t^{l'+m'}D^{l'}) = 0.$$

If $m = 0$, $m' \neq 0$, by Proposition 3.3 we have

$$\beta_1(t^lD^l, t^{l'+m'}D^{l'}) = -\frac{l'+m'}{l+1}\beta_1(t^{l+1}D^l, t^{l'+m'-1}D^{l'}) = 0.$$

Hence, we always have

$$\beta_1(t^{l+m}D^l, t^{l'+m'}D^{l'}) = 0, \qquad \text{if} \quad m \neq -m'.$$

Now we assume $m = -m'$, $0 \leqslant m + l \leqslant l + l'$, then

$$l' + m' = l' - m \geqslant 0.$$

By Proposition 3.3,

$$\begin{aligned}&\beta_1(t^{l+m}D^l, t^{l'-m}D^{l'})\\ &= \frac{-(l'-m)}{l+m+1}\cdot\frac{-(l'-m-1)}{l+m+2}\cdots\frac{0}{l+l'+1}\beta_1(t^{l+l'+1}D^l, t^{-1}D^{l'})\\ &= 0.\end{aligned}$$

This completes the proof.

LEMMA 3.6.

$$\text{(i)} \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \beta_1(t^{-1}, t^2D) \\ \beta_1(t^{-1}D, t^2) \end{pmatrix} = \begin{pmatrix} 2\beta_1(t, t^{-1}) \\ 3\beta_1(t, t^{-1}) \end{pmatrix}, \tag{3.4}$$

(ii) *if* $k \geqslant 2$,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -1 & 1 \\ 0 & 0 & \cdots & -1 & 0 & 1 \\ & \cdots & \cdots & & \cdots & \\ -1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & k\binom{2k-1}{k+1} & \cdots & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} \beta_1(t^{-1}, t^{k+1}D^k) \\ \beta_1(t^{-1}D, t^{k+1}D^{k-1}) \\ \vdots \\ \beta_1(t^{-1}D^{k-1}, t^{k+1}D) \\ \beta_1(t^{-1}D^k, t^{k+1}) \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{k-1} \\ b_k \end{pmatrix}. \tag{3.5}$$

where

$$b_u = \frac{1}{(k-u+1)(k+2)} \left(\sum_{j=0}^{k-u-1} \binom{k-u-1}{j} \right.$$
$$\times (-1) \cdots (-1-k+u+j)\, \beta_1(t^{-k+u+j-2}D^{u+j}, t^{k+2})$$
$$\left. - \sum_{i=0}^{k-u-1} \binom{k-u-1}{i} (k+2) \cdots (u+i+2)\, \beta_1(t^{u+i+1}D^i, t^{-1}D^u) \right),$$

$u = 0, ..., k-1$,

$$b_k = \frac{1}{k-1} \left(\sum_{i=0}^{k-2} \binom{k}{i} (k-1) \cdots i \beta_1(t^{k+i}D^i, t^{-k+1}D) \right.$$
$$\left. - \sum_{j=0}^{k-2} \binom{k}{j} (-k+1) \cdots (-2k+2+j)\, \beta_1(t^{-k+2+j}D^{j+1}, t^{k-1}) \right).$$

Proof. $[D^{l+1}, t^{l+l'+2}] = \sum_{i=0}^{l} \binom{l+1}{i} (l+l'+2) \cdots (l'+2+i)\, t^{l'+i+1}D^i$. We have

$$\beta_1([D^{l+1}, t^{l+l'+2}], t^{-1}D^{l'})$$
$$= \sum_{i=0}^{l} \binom{l+1}{i} (l+l'+2) \cdots (l'+2+i)\, \beta_1(t^{l'+i+1}D^i, t^{-1}D^{l'}).$$

On the other hand,

$$
\begin{aligned}
&\beta_1([D^{l+1}, t^{l+l'+2}], t^{-1}D^{l'})\\
&\quad = \beta_1([D^{l+1}, t^{-1}D^{l'}], t^{l+l'+2}) + \beta_1(D^{l+1}, [t^{l+l'+2}, t^{-1}D^{l'}])\\
&\quad = \beta_1\left(\sum_{j=0}^{l}\binom{l+1}{j}(-1)\cdots(-1-(l+1-j)+1)\right.\\
&\qquad \left.\times\, t^{-1-(l+1-j)}D^{l'+j}, t^{l+l'+2}\right)\\
&\quad = \sum_{j=0}^{l}\binom{l+1}{j}(-1)\cdots(-1-l+j)\\
&\qquad \times \beta_1(t^{-l+j-2}D^{l'+j}, t^{l+l'+2}) \qquad \text{(by Proposition 3.5).}
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\sum_{i=0}^{l}\binom{l+1}{i}(l+l'+2)\cdots(l'+2+i)\,\beta_1(t^{l'+i+1}D^{i}, t^{-1}D^{l'})\\
&\quad = \sum_{j=0}^{l}\binom{l+1}{j}(-1)\cdots(-1-l+j)\,\beta_1(t^{-l+j-2}D^{l'+j}, t^{l+l'+2}). \qquad (3.6)
\end{aligned}
$$

By Proposition 3.3,

$$
\beta_1(t^{-2}D^{l+l'}, t^{l+l'+2}) = (l+l'+2)\,\beta_1(t^{-1}D^{l+l'}, t^{l+l'+1}).
$$

Thus (3.6) can be rewritten as

$$
\begin{aligned}
&-\beta_1(t^{-1}D^{l'}, t^{l+l'+1}D^{l}) + \beta_1(t^{-1}D^{l+l'}, t^{l+l'+1})\\
&\quad = \frac{1}{(l+1)(l+l'+2)}\left(\sum_{j=0}^{l-1}\binom{l+1}{j}(-1)\cdots(-1-l+j)\right.\\
&\qquad \times \beta_1(t^{-l+j-2}D^{l'+j}, t^{l+l'+2})\\
&\qquad \left. -\sum_{i=0}^{l-1}\binom{l+1}{i}(l+l'+2)\cdots(l'+2+i)\,\beta_1(t^{l'+i+1}D^{i}, t^{-1}D^{l'})\right). \qquad (3.7)
\end{aligned}
$$

Setting $l'=0$, $l=1$ and using Proposition 3.3, we obtain

$$
-\beta_1(t^{-1}, t^2D) + \beta_1(t^{-1}D, t^2) = 2\beta_1(t^{-1}, t). \qquad (3.8)
$$

Since

$$
\begin{aligned}
\beta_1([tD^2, t^2], t^{-1}) &= \beta_1(4t^2D + 2t, t^{-1})\\
&= -4\beta_1(t^{-1}, t^2D) + 2\beta_1(t, t^{-1}),\\
\beta_1([tD^2, t^2], t^{-1}) &= \beta_1([tD^2, t^{-1}], t^2) + \beta_1(tD^2, [t^2, t^{-1}])\\
&= \beta_1(-2t^{-1}D + 2t^{-2}, t^2)\\
&= -2\beta_1(t^{-1}D, t^2) + 2\beta_1(t^{-2}, t^2).
\end{aligned}
$$

By Proposition 3.3, $\beta_1(t^{-2}, t^2) = 2\beta_1(t^{-1}, t)$, hence,

$$2\beta_1(t^{-1}, t^2D) - \beta_1(t^{-1}D, t^2) = 3\beta_1(t, t^{-1}). \tag{3.9}$$

From (3.8), (3.9), we have (3.4).

For $k \geqslant 2$,

$$\begin{aligned}&\beta_1([t^{k+1}D^k, t^{k-1}], t^{-k+1}D)\\ &\quad= \sum_{j=0}^{k-1} \binom{k}{j} (k-1)\cdots j\beta_1(t^{k+j}D^j, t^{-k+1}D).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\beta_1([t^{k+1}D^k, t^{k-1}], t^{-k+1}D)\\
&\quad= \beta_1([t^{k+1}D^k, t^{-k+1}D], t^{k-1}) + \beta_1(t^{k+1}D^k, [t^{k-1}, t^{-k+1}D])\\
&\quad= \beta_1\left(\sum_{i=0}^{k-1} \binom{k}{i} (-k+1)\cdots(-2k+2+i)\, t^{-k+2+i}D^{i+1}, t^{k-1}\right)\\
&\qquad - (k+1)\,\beta_1(tD^k, t^{k-1}) - (k-1)\,\beta_1(t^{k+1}D^k, t^{-1})\\
&\quad= k(-k+1)\,\beta_1(tD^k, t^{k-1}) - (k+1)\,\beta_1(tD^k, t^{k-1}) + (k-1)\\
&\qquad \times \beta_1(t^{-1}, t^{k+1}D^k)\\
&\qquad + \sum_{i=0}^{k-2} \binom{k}{i} (-k+1)\cdots(-2k+2+i)\,\beta_1(t^{-k+2+i}D^{i+1}, t^{k-1})\\
&\quad= (k-1)\,\beta_1(t^{-1}, t^{k+1}D^k)\\
&\qquad + \sum_{i=0}^{k-2} \binom{k}{i} (-k+1)\cdots(-2k+2+i)\,\beta_1(t^{-k+2+i}D^{i+1}, t^{k-1})\end{aligned}$$

(by Proposition 3.5).

We have

$$\begin{aligned}&(k-1)\,\beta_1(t^{-1}, t^{k+1}D^k) - k(k-1)\,\beta_1(t^{2k-1}D^{k-1}, t^{-k+1}D)\\
&\quad= \sum_{j=0}^{k-2} \binom{k}{j} (k-1)\cdots j\beta_1(t^{k+j}D^j, t^{-k+1}D)\\
&\qquad - \sum_{i=0}^{k-2} \binom{k}{i} (-k+1)\cdots(-2k+2+i)\,\beta_1(t^{-k+2+i}D^{i+1}, t^{k-1}).\end{aligned}$$

By Proposition 3.3,

$$\beta_1(t^{2k-1}D^{k-1}, t^{-k+1}D) = -\binom{2k-1}{k+1}\,\beta_1(t^{-1}D, t^{k+1}D^{k-1}).$$

We have

$$\beta_1(t^{-1}, t^{k+1}D^k) + k\binom{2k-1}{k+1}\beta_1(t^{-1}D, t^{k+1}D^{k-1})$$

$$= \frac{1}{k-1}\left(\sum_{j=0}^{k-2}\binom{k}{j}(k-1)\cdots j\beta_1(t^{k+j}D^j, t^{-k+1}D)\right.$$

$$\left. - \sum_{i=0}^{k-2}\binom{k}{i}(-k+1)\cdots(-2k+2+i)\,\beta_1(t^{-k+2+i}D^{i+1}, t^{k-1})\right). \quad (3.10)$$

Taking $k = l + l'$ in (3.7), we obtain (3.5) from (3.7) and (3.10).

Proof of Theorem 2.1. Let β be a 2-cocycle on $\mathfrak{a}$, then there exists a 2-cocycle β_1 on $\mathfrak{a}$, β_1 is equivalent to β, and β_1 satisfies (3.2), (3.3). If $\beta_1 = 0$, this implies $\beta_1 = 0\alpha$, $0 \in \mathbb{C}$.

If $\beta_1 \neq 0$, i.e., $\beta_1(t^{l+m}D^l, t^{l'-m}D^{l'}) \neq 0$ for some $l \geqslant 0$, $l' \geqslant 0$, $m \in \mathbb{Z}$. We conclude that $\beta_1(t, t^{-1}) \neq 0$. Otherwise, $\beta_1(t, t^{-1}) = 0$. By Propositions 3.3 and 3.5, $\beta_1(t^m, t^n) = 0$ for $m, n \in \mathbb{Z}$. Now we prove that

$$\beta_1(t^{l+m}D^l, t^{l'+m'}D^{l'}) = 0 \quad (3.11)$$

by induction on $l + l'$. We have known that (3.11) holds for $l + l' = 0$. Assume that (3.11) holds for $l + l' \leqslant n$. If $l + l' = n + 1$, since the coefficient matrices in (3.4), (3.5) are all nondegenerate and the right hand of (3.4) or (3.5) for $k = n + 1$ is zero, we have

$$\beta_1(t^{-1}D^l, t^{n+2}D^{l'}) = 0,$$

$l + l' = n + 1$. By Propositions 3.3 and 3.5, we see that (3.11) holds for $l + l' = n + 1$. Equation (3.11) implies $\beta_1 = 0$. This is a contradiction. Hence we have $\beta_1(t, t^{-1}) \neq 0$.

Denote $a = \beta_1(t, t^{-1}) \in \mathbb{C}^*$. Define $\beta_2 = a^{-1}\beta_1$, then β_2 is a 2-cocycle and β_2 satisfies (3.2), (3.3). All results for β_1 hold for β_2 and α. Now we prove that $\beta_2 = \alpha$.

By definition, we have $\beta_2(t, t^{-1}) = \alpha(t, t^{-1}) = 1$. From Propositions 3.3 and 3.5, $\beta_2(t^m, t^n) = \alpha(t^m, t^n)$, $m, n \in \mathbb{Z}$. Assume $\beta_2(t^{l+m}D^l, t^{l'+m'}D^{l'}) = \alpha(t^{l+m}D^l, t^{l'+m'}D^{l'})$ for $l + l' \leqslant n$, if $l + l' = n + 1$. Replace β_1 by β_2 and α, respectively, and set $k = n + 1$; since the coefficient matrices in (3.4), (3.5) are nondegenerate, we have

$$\beta_2(t^{-1}D^l, t^{n+2}D^{l'}) = \alpha(t^{-1}D^l, t^{n+2}D^{l'}),$$

$l + l' = n + 1$. This implies $\beta_2 = \alpha$. Thus we always have $\beta_1 = a\alpha$, $a \in \mathbb{C}$, and β is equivalent to $a\alpha$, $a \in \mathbb{C}$. This completes the proof of the theorem.

4. Proof of Theorem 2.2

Let e_{ij} be the $n \times n$ matrix which is 1 in the i, j-entry and 0 everywhere else, then $\mathfrak{A}$ has a basis [10]

$$\{t^{l+m}D^l \otimes e_{ij} \mid l \geqslant 0, m \in \mathbb{Z}, 1 \leqslant i, j \leqslant n\},$$

and

$$\begin{aligned}
&[t^{l+m}D^l \otimes e_{ij}, t^{l'+m'}D^{l'} \otimes e_{i'j'}] \\
&\quad = \delta_{j,i'}(t^{l+m}D^l) \circ (t^{l'+m'}D^{l'}) \\
&\qquad \otimes e_{ij'} - \delta_{j',i}(t^{l'+m'}D^{l'}) \circ (t^{l+m}D^l) \otimes e_{i'j} \\
&\quad = \delta_{j,i'} \sum_{p=0}^{l} \binom{l}{p} t^{l+m}(D^{l-p}(t^{l'+m'}))\, D^{l'+p} \otimes e_{ij'} \\
&\qquad - \delta_{j',i} \sum_{q=0}^{l'} \binom{l'}{q} t^{l'+m'}(D^{l'-q}(t^{l+m}))\, D^{l+q} \otimes e_{i'j}.
\end{aligned}$$

In particular,

$$\begin{aligned}
[D \otimes e_{ii}, t^{l+m}D^l \otimes e_{ii}] &= [D, t^{l+m}D^l] \otimes e_{ii} \\
&= (l+m)\, t^{l+m-1}D^l \otimes e_{ii}, \qquad (4.1)
\end{aligned}$$

$$\begin{aligned}
[t^{-1}D^l \otimes e_{ii}, tD \otimes e_{ii}] &= [t^{-1}D^l, tD] \otimes e_{ii} \\
&= (l+1)\, t^{-1}D^l \otimes e_{ii}, \qquad (4.2)
\end{aligned}$$

$$[1 \otimes e_{ii}, t^{l+m}D^l \otimes e_{ij}] = t^{l+m}D^l \otimes e_{ij}, \qquad i \neq j. \qquad (4.3)$$

From the definition (2.5),

$$\begin{aligned}
&\psi(t^{l+m}D^l \otimes e_{ij}, t^{l'+m'}D^{l'} \otimes e_{i'j'}) = 0 \\
&\quad \Leftrightarrow \delta_{i,j'}\delta_{i',j}\delta_{m,-m'} = 0 \qquad \text{or} \qquad \delta_{i,j'}\delta_{i',j}\delta_{m,-m'} \neq 0, \quad 0 \leqslant m+l \leqslant l+l' \qquad (4.4)
\end{aligned}$$

and

$$\psi(t^{l+m}D^l \otimes e_{ij}, t^{l'+m'}D^{l'} \otimes e_{ji}) = \alpha(t^{l+m}D^l, t^{l'+m'}D^{l'}). \qquad (4.5)$$

Let φ be a 2-cocycle on $\mathfrak{A}$, and we define a linear function f_φ on $\mathfrak{A}$:

$$f_\varphi(t^{l+m-1}D^l \otimes e_{ii}) = \frac{1}{l+m}\, \varphi(D \otimes e_{ii}, t^{l+m}D^l \otimes e_{ii}), \qquad l+m \neq 0,$$

$$f_\varphi(t^{-1}D^l \otimes e_{ii}) = \frac{1}{l+1}\, \varphi(t^{-1}D^l \otimes e_{ii}, tD \otimes e_{ii}),$$

$$f_\varphi(t^{l+m}D^l \otimes e_{ij}) = \varphi(1 \otimes e_{ii}, t^{l+m}D^l \otimes e_{ij}), \qquad i \neq j.$$

Then $\varphi_1 = \varphi - \alpha_{f_\varphi}$ is a 2-cocycle on $\mathfrak{A}$ and φ_1 is equivalent to φ, where α_{f_φ} is the trivial 2-cocycle induced by f_φ. From (4.1), (4.2), (4.3) we have

$$\varphi_1(D \otimes e_{ii}, t^{l+m}D^l \otimes e_{ii}) = 0, \qquad l+m \neq 0, \tag{4.6}$$

$$\varphi_1(t^{-1}D^l \otimes e_{ii}, tD \otimes e_{ii}) = 0, \tag{4.7}$$

$$\varphi_1(1 \otimes e_{ii}, t^{l+m}D^l \otimes e_{ij}) = 0, \qquad i \neq j. \tag{4.8}$$

Comparing (4.6), (4.7) with (3.2), (3.3), from the proof of Theorem 2.1 we get immediately.

LEMMA 4.1.

$$\begin{aligned}&\varphi_1(t^{l+m}D^l \otimes e_{ii}, t^{l'+m'}D^{l'} \otimes e_{ii})\\ &\quad = a_i \alpha(t^{l+m}D^l, t^{l'+m'}D^{l'}), \qquad a_i \in \mathbb{C}.\end{aligned}$$

LEMMA 4.2. $\varphi_1(t^{l+m}D^l \otimes e_{ii}, t^{l'+m'}D^{l'} \otimes e_{ij}) = 0$, $i \neq j$.

Proof. First we prove that $\varphi_1(1 \otimes e_{ii}, t^{l+m}D^l \otimes e_{ji}) = 0$ for $i \neq j$.

$$\begin{aligned}&\varphi_1(1 \otimes e_{ii}, t^{l+m}D^l \otimes e_{ji})\\ &\quad = \varphi_1(1 \otimes e_{ii}, [1 \otimes e_{jj}, t^{l+m}D^l \otimes e_{ji}])\\ &\quad = \varphi_1([1 \otimes e_{ii}, 1 \otimes e_{jj}], t^{l+m}D^l \otimes e_{ji})\\ &\qquad + \varphi_1(1 \otimes e_{jj}, [1 \otimes e_{ii}, t^{l+m}D^l \otimes e_{ji}])\\ &\quad = \varphi_1(1 \otimes e_{jj}, -t^{l+m}D^l \otimes e_{ji}).\end{aligned}$$

From (4.8) we have that the last term is zero.

If $i \neq j$, we have

$$\begin{aligned}&\varphi_1(t^{l+m}D^l \otimes e_{ii}, t^{l'+m'}D^{l'} \otimes e_{ij})\\ &\quad = \varphi_1(t^{l+m}D^l \otimes e_{ii}, [t^{l'+m'}D^{l'} \otimes e_{ij}, 1 \otimes e_{jj}])\\ &\quad = \varphi_1([t^{l+m}D^l \otimes e_{ii}, t^{l'+m'}D^{l'} \otimes e_{ij}], 1 \otimes e_{jj})\\ &\qquad + \varphi_1(t^{l'+m'}D^{l'} \otimes e_{ij}, [t^{l+m}D^l \otimes e_{ii}, 1 \otimes e_{jj}])\\ &\quad = \varphi_1((t^{l+m}D^l) \circ (t^{l'+m'}D^{l'}) \otimes e_{ij}, 1 \otimes e_{jj})\\ &\quad = 0.\end{aligned}$$

LEMMA 4.3. $\varphi_1(f \otimes e_{ij}, g \otimes e_{jk}) = 0$, $i \neq j$, $j \neq k$, $k \neq i$, $f, g \in \mathfrak{a}$.

Proof.

$$
\begin{aligned}
&\varphi_1(f\otimes e_{ij}, g\otimes e_{jk})\\
&\quad=\varphi_1([1\otimes e_{ii}, f\otimes e_{ij}], g\otimes e_{jk})\\
&\quad=\varphi_1([1\otimes e_{ii}, g\otimes e_{jk}], f\otimes e_{ij})+\varphi_1(1\otimes e_{ii}, [f\otimes e_{ij}, g\otimes e_{jk}])\\
&\quad=\varphi_1(1\otimes e_{ii}, (f\circ g)\otimes e_{ik})\\
&\quad=0 \qquad \text{(by Lemma 4.2).}
\end{aligned}
$$

LEMMA 4.4. $\varphi_1(f\otimes e_{ij}, g\otimes e_{sk})=0$, $i\neq k$, $j\neq s$, f, $g\in\mathfrak{a}$.

Proof. If $i\neq j$,

$$
\begin{aligned}
&\varphi_1(f\otimes e_{ij}, g\otimes e_{sk})\\
&\quad=\varphi_1([1\otimes e_{ii}, f\otimes e_{ij}], g\otimes e_{sk})\\
&\quad=\varphi_1(1\otimes e_{ii}, [f\otimes e_{ij}, g\otimes e_{sk}])+\varphi_1([1\otimes e_{ii}, g\otimes e_{sk}], f\otimes e_{ij})\\
&\quad=\delta_{i,s}\varphi_1(g\otimes e_{ik}, f\otimes e_{ij})\\
&\quad=-\delta_{i,s}\varphi_1(f\otimes e_{ij}, g\otimes e_{sk}).
\end{aligned}
$$

Hence we have $\varphi_1(f\otimes e_{ij}, g\otimes e_{sk})=0$.

Since $[\mathfrak{a},\mathfrak{a}]=\mathfrak{a}$, $\forall f\in\mathfrak{a}$, f can be rewritten as

$$\sum_{r=1}^{u}[f_{1r}, f_{2r}], \qquad f_{1r}, f_{2r}\in\mathfrak{a}.$$

Thus, if $i=j$,

$$
\begin{aligned}
&\varphi_1(f\otimes e_{ii}, g\otimes e_{sk})\\
&\quad=\varphi_1\left(\sum_{r=1}^{u}[f_{1r}\otimes e_{ii}, f_{2r}\otimes e_{ii}], g\otimes e_{sk}\right)\\
&\quad=\sum_{r=1}^{u}\varphi_1([f_{1r}\otimes e_{ii}, g\otimes e_{sk}], f_{2r}\otimes e_{ii})\\
&\qquad+\sum_{r=1}^{u}\varphi_1(f_{1r}\otimes e_{ii}, [f_{2r}\otimes e_{ii}, g\otimes e_{sk}])\\
&\quad=0.
\end{aligned}
$$

LEMMA 4.5. *a_i's in Lemma* 4.1 *are equal.*

Proof. We prove that if $i \neq j$, $a_i = a_j$. Since

$$\begin{aligned}
&\varphi_1(t\otimes e_{ii}, t^{-1}\otimes e_{ii})\\
&\quad = \varphi_1(t\otimes(e_{ii}-e_{jj}), t^{-1}\otimes e_{ii}) \qquad \text{(by Lemma 4.4)}\\
&\quad = \varphi_1([1\otimes e_{ij}, t\otimes e_{ji}], t^{-1}\otimes e_{ii})\\
&\quad = \varphi_1(1\otimes e_{ij}, [t\otimes e_{ji}, t^{-1}\otimes e_{ii}]) + \varphi_1([1\otimes e_{ij}, t^{-1}\otimes e_{ii}], t\otimes e_{ji})\\
&\quad = \varphi_1(1\otimes e_{ij}, 1\otimes e_{ji}) + \varphi_1(-t^{-1}\otimes e_{ij}, t\otimes e_{ji}),\\
&\varphi_1(t\otimes e_{jj}, t^{-1}\otimes e_{jj})\\
&\quad = \varphi_1(t\otimes e_{jj}, t^{-1}\otimes(e_{jj}-e_{ii})) \qquad \text{(by Lemma 4.4)}\\
&\quad = \varphi_1(t\otimes e_{jj}, [1\otimes e_{ji}, t^{-1}\otimes e_{ij}])\\
&\quad = \varphi_1([t\otimes e_{jj}, 1\otimes e_{ji}], t^{-1}\otimes e_{ij})\\
&\qquad + \varphi_1(1\otimes e_{ji}, [t\otimes e_{jj}, t^{-1}\otimes e_{ij}])\\
&\quad = \varphi_1(t\otimes e_{ji}, t^{-1}\otimes e_{ij}) + \varphi_1(1\otimes e_{ji}, -1\otimes e_{ij}).
\end{aligned}$$

This implies $\varphi_1(t\otimes e_{ii}, t^{-1}\otimes e_{ii}) = \varphi_1(t\otimes e_{jj}, t^{-1}\otimes e_{jj})$, that is, $a_i = a_j$.

In order to prove $\varphi_1(f\otimes e_{ii}, g\otimes e_{ii}) = \varphi_1(f\otimes e_{ij}, g\otimes e_{ji})$, $i \neq j$, $f, g \in \mathfrak{a}$, we need the following fact.

LEMMA 4.6. $\varphi_1(1\otimes e_{ij}, f\otimes e_{ji}) = \varphi_1(f\otimes e_{ij}, 1\otimes e_{ji})$, $i \neq j$, $f \in \mathfrak{a}$.

Proof.

$$\begin{aligned}
&\varphi_1(1\otimes e_{ij}, f\otimes e_{ji})\\
&\quad = \varphi_1(1\otimes e_{ij}, [1\otimes e_{ji}, f\otimes e_{ii}])\\
&\quad = \varphi_1([1\otimes e_{ij}, 1\otimes e_{ji}], f\otimes e_{ii}) + \varphi_1(1\otimes e_{ji}, [1\otimes e_{ij}, f\otimes e_{ii}])\\
&\quad = \varphi_1(1\otimes e_{ii}, f\otimes e_{ii}) + \varphi_1(1\otimes e_{ji}, -f\otimes e_{ij}) \qquad \text{(by Lemma 4.4)}\\
&\quad = \varphi_1(f\otimes e_{ij}, 1\otimes e_{ji}). \qquad \text{(by Lemma 4.1)}.
\end{aligned}$$

LEMMA 4.7. $\varphi_1(f\otimes e_{ii}, g\otimes e_{ii}) = \varphi_1(f\otimes e_{ij}, g\otimes e_{ji})$, $i \neq j$, $f, g \in \mathfrak{a}$.

Proof.

$$\begin{aligned}
&\varphi_1(f\otimes e_{ii}, g\otimes e_{ii})\\
&\quad = \varphi_1([f\otimes e_{ij}, 1\otimes e_{ji}], g\otimes e_{ii}) \qquad \text{(by Lemma 4.4)}\\
&\quad = \varphi_1(f\otimes e_{ij}, [1\otimes e_{ji}, g\otimes e_{ii}])\\
&\qquad + \varphi_1([f\otimes e_{ij}, g\otimes e_{ii}], 1\otimes e_{ji})\\
&\quad = \varphi_1(f\otimes e_{ij}, g\otimes e_{ji}) + \varphi_1((-g\circ f)\otimes e_{ij}, 1\otimes e_{ji}). \qquad (4.9)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\varphi_1(f\otimes e_{ii}, g\otimes e_{ii})\\
&\quad=\varphi_1(f\otimes e_{jj}, g\otimes e_{jj}) \qquad \text{(by Lemma 4.5)}\\
&\quad=\varphi_1(f\otimes e_{jj}, [g\otimes e_{ji}, 1\otimes e_{ij}]) \qquad \text{(by Lemma 4.4)}\\
&\quad=\varphi_1([f\otimes e_{jj}, g\otimes e_{ji}], 1\otimes e_{ij})\\
&\qquad+\varphi_1(g\otimes e_{ji}, [f\otimes e_{jj}, 1\otimes e_{ij}])\\
&\quad=\varphi_1((f\circ g)\otimes e_{ji}, 1\otimes e_{ij})+\varphi_1(f\otimes e_{ij}, g\otimes e_{ji}).
\end{aligned}\tag{4.10}$$

From (4.9), (4.10), we have

$$\begin{aligned}
&\varphi_1((f\circ g)\otimes e_{ji}, 1\otimes e_{ij})\\
&\quad=\varphi_1(1\otimes e_{ji}, (g\circ f)\otimes e_{ij})\\
&\quad=\varphi_1((g\circ f)\otimes e_{ji}, 1\otimes e_{ij}) \qquad \text{(by Lemma 4.6)},
\end{aligned}$$

that is,

$$\varphi_1([f, g]\otimes e_{ji}, 1\otimes e_{ij})=0.$$

Since $[\mathfrak{a}, \mathfrak{a}]=\mathfrak{a}$, we have $\varphi_1(\mathfrak{a}\otimes e_{ji}, 1\otimes e_{ij})=0$. Thus (4.9) becomes

$$\varphi_1(f\otimes e_{ii}, g\otimes e_{ii})=\varphi_1(f\otimes e_{ij}, g\otimes e_{ji}).$$

From Lemma 4.1 to Lemma 4.7, we can see easily that $\varphi_1=a\psi$ for some $a\in\mathbb{C}$. This completes the proof of Theorem 2.2.

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